HABILITATION THESIS

New results in the theory of Countable Iterated Function Systems

- abstract -

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Abstract

The goal of this thesis is to describe the contributions of the author in the theory of infinite iterated function systems which generalizes the classical Hutchinson-Barnsley theory of (finite) iterated function systems. These contributions are published in several hight impact international journals and represent new results achieved by the author after obtaining the Ph.D. title in 2001.

Theory of fractals is now widely used in the modern approach in almost all areas of science. Deterministic and random fractals, within the framework of Iterated Function Systems (IFS), have been used to model and study a wide range of phenomena across many areas of science and technology: statistical physics, neuronal nets, image compression, natural sciences, learning automata and computer graphics. E.g., the use of fractals in image processing may give a considerable compression of relevant data (see for instance Senesi N., Wilkinson K.J. (2008) have been investigated how the fractal approach has been applied in order to understand the structure and reactivity of natural, environmental systems). The mathematics of iterated functions is particularly appealing and it is embedded in the general theory of dynamical systems.

IFSs are among the basic methods for generating fractals. The term itself was introduced in 1985 by Michael Fielding Barnsley and Stephen Demko (*Iterated function systems and the global construction of fractals*, Proc. R. Soc. Lond. Ser. A **399** (1985), 243-275), but the essential concept is usually attributed to John E. Hutchinson (*Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713-747). Edward R. Vrscay (1990) traces the idea further back to R.F. Williams (1971), who studied fixed points of finite compositions of contractions. A theory of (self-similar) fractal sets and measures as well as the concept of IFS were also developed by Masayoshi Hata (1985). Existence, convergence and uniqueness results were based on contraction properties of the fractal operator with respect to the Pompeiu-Hausdorff metric for sets and the Monge-Kantorovich metric for measures. In fact, in his famous paper, Hutchinson proved that, given an IFS, i.e. a set of self contractions $(\omega_n)_{n=1}^N$ on a complete metric space X, the set mapping $\mathcal{S}_N : \mathcal{K}(X) \to \mathcal{K}(X)$, $\mathcal{S}_N(B) = \bigcup\limits_{n=1}^{N} \omega_n(B)$ is a contraction on the hyperspace of all nonempty compact subsets of X endowed with Pompeiu-Hausdorff metric (this space being complete too). His unique fixed point $A \in \mathcal{K}(X)$ is called the attractor of the IFS. A is generally a fractal set.

Last decades many authors had to deal with various extensions of Barnsley-Hutchinson's classical framework for fractals to more general spaces, generalized contractions and infinite IFSs or, moreover, to multifunction systems. For infinite iterated function systems there are the remarkable works of Henning Fernau, Krzysztof Leśniak, Franklin Mendivil, Mariusz Urbanski, R. Daniel Mauldin and others.

In a proper manner, the Countable Iterated Function System (shortly CIFS) has been described in: N.A. Secelean, *Countable Iterated Function Systems*, Far East J. Dyn. Syst. **3**(2) (2001), 149-167 and some of its properties can be found in some papers of the same author (2001,2002,2003).

Let us mention that another way to generate fractals is due to Gaston Julia and Pierre Fatou who studied the iterations of rational functions on the Riemann sphere. It is not the goal of the present thesis to discuss this topic.

The thesis is organized in six chapters, the last one being dedicated to description of some future plans regarding the professional and scientific career of the author. At the beginning of each of the first five chapters and of certain sections or subsections there is a brief description of the content together with the corresponding references. All sections of these chapters, except those devoted to introduction and to preliminary facts, contain results obtained and published by the author of this thesis after obtaining the PhD title, the references to the respective publications (citations) being revealed by bold face. Though the proofs of almost all these results are given while, for simplicity, the proofs of some theorems were omitted and the citation sources are specified.

Chapter 1 has two purposes. The first one is to describe some historical aspects concerning the Hutchinson-Barnsley theory of (finite) IFS together with some concepts, notations and main properties used in that theory such as Pompeiu-Hausdorff metric, fractal operator, Monge-Kantorovich metric. It is presented an extension of IFS to countable many self contractions $(\omega_n)_{n\geq 1}$ on a compact metric space X. The Hutchinson operator will be $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$, $\mathcal{S}(B) = \overline{\bigcup_{n\geq 1} \omega_n(B)}$, where $\mathcal{K}(X)$ denotes the class of all nonempty compact subsets of X. The main properties of CIFS concerning the attractor, Hutchinson measure, continuous dependence of parameter are described.

The second goal of this chapter (see *Section* 1.3) is a survey of IFSs and CIFSs in some general settings. Thus, the existence and uniqueness of a set fixed point (invariant) set of an IFS and, resp. a CIFS, composed of continuous self mappings on a topological space are investigated. These results are very useful in order to obtain, as particular cases, new generalizations of the classical CIFSs (see Section 3.3).

Chapter 2 is intended to describe some contributions of the author concerning the generalization of the theory of fractal interpolation from the finite set of data (as it was initially developed by Barnsley) to the countable setting.

In *Section* 2.2, a compact and pathwise connected metric space Y and a countable system of data $\Delta = (x_n, y_n)_{n\geq 0} = \big\{(x_n, y_n) \in X : \ n = 0, 1, \dots\big\}$, where $X = [a, b] \times Y$, are considered. It is showed that, under some suitable assumptions, there is a fractal interpolation function f (i.e. a continuous map $f : [a, b] \to Y$ such that $f(x_n) = y_n$, $n \geq 0$) those graphic is the attractor A of some proper CIFS (see Theorem 2.2.2). Moreover, an approximation of A is given (see Corollary 2.2.1).

In *Notes on Fractal Interpolation*, Novi Sad J. Math., **30** (2000), no. 3, 59-68, Lj. M. Kocić and A.C. Simoncelli described the affine invariance of an interpolatory scheme for a finite system of data in $\mathbb{R}^2$. In *Section* 2.3 we extend this study to the case of countable set of data. The main result is that, given a countable system of data Δ in $[a, b] \times Y$, where $[a, b]$ is a real interval, Y is a compact and arcwise connected metric space and ψ is a proper continuous function which transforms Δ in yet another countable system of data, then $\psi(A)$ is the attractor corresponding to $\psi(\Delta)$, where A is the attractor associated to Δ (see Theorem 2.3.1). The particular case $\Delta \subset \mathbb{R}^2$ space is investigated.

In *Section* 2.4 we consider two metric spaces P, Y, the second one being compact and arcwise connected and a countable system of data $\Delta(t) := \big(x_n(t), y_n(t)\big)_{n\geq 0} \subset [a, b] \times Y$, where, for each $n \geq 0$, $x_n : P \to \mathbb{R}$, $y_n : P \to Y$ are maps that satisfy the following conditions: $(x_n)_n$ is strictly increasing, $(y_n)_n$ is pointwise convergent and there are $a, b \in \mathbb{R}$ such that $a = x_0(t)$, $b = \lim_n x_n(t)$ for all $t \in P$. The main result can be found in Theorem 2.4.3 which states that, under some certain assumptions, the associated fractal interpolation function depends continuously with respect to the parameter $t \in P$.

Chapter 3 is devoted to the contributions of the author to the current effort of many researchers to extend the Hutchinson-Barnsley theory by considering IFS consisting of generalized contractions instead of Banach contractions. In *Section* 3.2 some mappings that satisfy some contractive type conditions together with their fixed point theorems are recalled. After that, in *Section* 3.3, we consider IFSs and CIFSs composed of, respectively, contractive maps, φ-contractions, Meir-Keeler type maps, F-contractions on a metric space into itself. Some sufficient conditions for the existence of a set fixed point or, respectively, of the attractor for each type of CIFS are given in Corollary 3.3.1, resp. Theorem 3.3.5 (for CIFS composed by φ_n-contractions), in Theorem 3.3.3, resp. Theorem 3.3.6 (for CIFS composed by Meir-Keeler type maps), in Theorem 3.3.2, resp. Theorem 3.3.4 (CIFS composed by contractive maps). The existence of the attractor of an IFS consisting of F-contractions and its successively approximation is proved in Theorem 3.3.7.

In **Chapter 4** other generalizations of the classical CIFS obtained recently by the author are provided. We started from the concept of Generalized Iterated Function System (GIFS) introduced by A. Mihail and R. Miculescu (*Applications of Fixed Point Theorems in the Theory of Generalized IFS*, Fixed Point Theory Appl., **2008**, Article ID 312876) as an IFS consisting of contractions on the product space $X^m = X \times \cdots \times X$, $m \geq 1$, into X, where X is a given metric space and X^m is endowed with the max metric. In *Section* 4.2 we remind the main results concerning the attractor and its approximation.

One of the most important result in the study of topological properties of the attractor of some IFS is given by M. Yamaguti, M. Hata, J. Kigami (1997) which established when this attractor is a connected set.

The aim of *Section* 4.3 is to give necessary and sufficient conditions for the connectedness of the attractor of a GIFS by extending the result of M. Yamaguti et all.

The theory of (finite) GIFS on a finite product X^m, where $m \geq 1$ and X is a complete metric space, is extended to the countable many contractions on X^m into X, where, in addition, X is compact. Thus, in *Section* 4.4 we define the Generalized Countable Iterated Function System (GCIFS) and describe new results obtained by author concerning its properties. The fact that any CIFS is a particular case of a GCIFS is revealed in Remark 4.4.1. It is showed that any GCIFS $(\omega_n)_n$ has an attractor which, under some suitable conditions, is the closure of the set of all fixed points of contractions $\widetilde{\omega}_{i_1} \circ \cdots \circ \widetilde{\omega}_{i_p}$, where $\widetilde{\omega_n}(x) := \omega_n(x, x, \ldots, x)$, $x \in X$, $n, i_1, \ldots, i_p \in \mathbb{N}$ (see Proposition 4.4.1).

In *Subsection* 4.4.1 we show that, whenever the contractions of a GCIFS are Lipschitz maps with respect to a parameter and the supremum of the Lipschitz constants is finite, then the attractor depends uniform continuously on the respective parameter (see Theorem 4.4.2).

Some ways to approximate the attractor of a GCIFS are presented in *Subsection*

4.4.2 (see Theorem 4.4.3 and Lemmas 4.4.4, 4.4.5). Notice that the attractor can be approximated by finite sets, this fact being very useful to its graphical representation by computer.

In *Subsection* 4.4.3 the Hutchinson measure associated to a GCIFS with probabilities (GCIFSp) is defined. We show that, if all contractions $(\omega_n)_n$ of a GCIFSp satisfy a proper Lipschitz type condition, then there is a unique normalized Borel measure which is invariant with respect to the Markov operator called Hutchinson measure associated to the respective GCIFSp (see Theorem 4.4.6). This measure is the limit with respect to the Monge-Kantorovich metric of the sequence of Hutchinson measures associated to the partial GIFSps (see Theorem 4.4.7). Moreover, the support of the associated Hutchinson measure is precisely the attractor of the respective GCIFS.

Section 4.5 is dedicated to the presentation of new results obtained very recently by author in order to improve the theory of GIFS and GCIFS in more general settings. In this respect we consider the GIFS and GCIFS composed of generalized contractions on the space $X^I := \{u = (x_i)_{i\in I};\ x_i \in X,\ \sup_{i,j\in I} \mathrm{d}(x_i, x_j) < \infty\}$, where (X, d) is a metric space and $I \subset \mathbb{N}$ is a nonempty set. That is $X^I = X^m = \big\{u = (x_1, \ldots, x_m);\ x_i \in X,\ i = 1, \ldots, m\big\}$, where $m = \mathrm{card} I$ if I is finite and, respectively, X^I is the $l^\infty(X)$ space of all bounded sequences in X when I is infinite. The space X^I is endowed with the sup metric.

In *Subsection* 4.5.1 are proved fixed point theorems for some generalized contractions $f : X^I \to X$, namely for φ-contractions (see Theorem 4.5.1), for contractive mappings (see Theorem 4.5.2) and for Meir-Keeler type maps (see Theorem 4.5.3). It is showed that each of the above mentioned functions has a unique fixed point which is approximated by the iterative sequence $(y_k)_{k\geq 0}$ associated with f at u defined by $y_0 = f(u)$, $y_k = f\big(\widetilde{f}^k(x_1), \widetilde{f}^k(x_2), \ldots\big)$, for every $k \geq 1$ and $u = (x_i)_{i\in I} \in X^I$, where $\widetilde{f} : X \to X$, $\widetilde{f}(x) = f(x, x, \ldots)$.

In *Subsection* 4.5.2 we introduce the Pompeiu-Hausdorff metric on $\mathcal{K}(X)^I$ by $H(A, B) := \sup_{i\in I} h(A_i, B_i)$ for all $A = (A_i)_{i\in I}, B = (B_i)_{i\in I} \in \mathcal{K}(X)^I$, where $\mathcal{K}(X)^I$ denotes the class of all sequences of sets $(K_i)_{i\in I}$, $K_i \in \mathcal{K}(X)$ and $\sup_{i\in I} \mathrm{diam}(K_i) < \infty$. It is proved that some contractivity properties of the functions $\omega : X^I \to X$ can be found again for the set function $A \mapsto \overline{\omega(A)}$ or, respectively, for $A \mapsto \omega(A)$ on $\mathcal{K}(X)^I$ (see Theorems 4.5.4, 4.5.5 and 4.5.6).

The main results of this section are given in *Subsection* 4.5.3 where GIFS and GCIFS consisting of φ-contractions, contractive maps and Meir-Keeler type contractions, respectively, are considered. Some sufficient conditions for the existence and uniqueness of the attractor for each such GIFS and GCIFS are given. Furthermore, some approximations of those attractors are also described.

Some significant particular cases in which the space X^I is equipped with other metrics are investigated in *Subsection* 4.5.4.

Chapter 5 is devoted to another extension on the Hutchinson-Barnsley theory of IFS by considering the TVS-cone metric spaces instead of metric spaces (TVS means topological vector space). By improving some results of Huang and Zhang (2007) and others, Wei-Shih Du (2010) showed that, if (X, ρ) is a TVS-cone metric space, then $d_\rho = \xi_e \circ \rho$ is a metric on X, ξ_e being a certain scalarization function. He proved the equivalence of vectorial versions of fixed point theorems in generalized cone metric spaces and scalar versions of the Banach contraction principle in metric spaces (in usual sense) for Banach contractions.

After some preliminary facts in a TVS-cone metric space X described in *Section* 5.1, we define in *Section* 5.2 a vector comparison operator $\varphi : K \to K$, where K is the cone from the definition of X and prove some of its properties. In Theorem 5.2.1 we show how a scalar comparison function ψ (in the sense from Section 3.2) can be associated to φ. Next, the φ-contractive operator T on a TVS-cone metric space (X, ρ) is defined and it is proved that it is a ψ-contraction in the metric space (X, d_ρ) (see Theorem 5.2.2). If, further, (X, ρ) is complete, then T is a Picard operator. We notice that the results of Huang, Zhang and Du can be obtained as particular cases of the aforesaid.

In *Section* 5.3 one can find an investigation of IFSs and CIFSs consisting of φ-operators on a TVS-cone metric space. First, we prove that the topologies associated to respectively (X, ρ) and (X, d_ρ) coincide (see Proposition 5.3.1). Next, if $\omega_n : X \to X$, $n = 1, \ldots, N$, are φ-contractions and (X, ρ) is a complete TVS-cone metric space, then the IFS $(\omega_n)_{n=1}^N$ has an attractor which is approximated by $\big(\mathcal{S}^p(B)\big)_p$ for all $B \in \mathcal{K}(X)$, where $\mathcal{S}$ is the Hutchinson operator, the limit being taken with respect to Pompeiu-Hausdorff metric defined with the aim of a certain metric (see Theorem 5.3.1). Theorem 5.3.2 provides a similar result for a CIFS composed of φ-operators on a compact TVS-cone metric space.

Chapter 6 is dedicated to presentation of some future plans regarding the professional and scientific career of the author. It describes the following.

1) The research directions that the author intends to follow.

A. Short term research projects:

A.1. *Generalized F-Iterated Function Systems on a product of metric spaces* where the author intends to investigate IFS and CIFS composed of F-contractions on the space X^I, $\emptyset \neq I \subset \mathbb{N}$, into X, where X is a given complete metric space.

A.2. *Weak F-contractions on partially ordered metric spaces* in which the purpose of the author is to describe a new class of contractive mappings (namely weak F-contractions) together with their fixed point properties, a fixed point theorem of weak F-contractions on a partially ordered metric space and an application to IFS theory.

A.3.*IFS and CIFS consisting of quasi-nonexpansive operators* where we intend to construct a class of IFSs composed of discontinuous Picard mappings.

A.4. *A sufficient condition for a CIFS consisting of continuous functions to be treated as one composed of contractive mappings*

A.5. *IFS composed of contractive mappings of integral type* where we intend to define F-contractions of integral type and to obtain new class of IFSs, resp. CIFSs, composed of such functions.

B. Long time research projects:

Some ideas to a future research work in this topic are mentioned.

2) Books: I intend to publish three books including two new editions.

3) Teaching: my purpose is to teach the course "Measure and fractals" and to propose a new course at the Master level containing countable iterated function systems and applications. I will intensify collaboration with universities in Romania and abroad. I apply to the rank of Professor at Lucian Blaga University of Sibiu.